Influence of distractors on student's performance and confidence in single choice quizzes

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Abstract

Single choice questions are a substantial part of exams in various fields, in particular in mathematics. However, few studies have examined how the provided incorrect answer options (the so-called 'distractors') in mathematical questions are affecting the achievement of the students. In this paper we describe a study designed to investigate not only the effect of distractors on the performance, but also which kind of distractors make students uncertain or, on the contrary, lull students into a false sense of security.

Introduction

As the number of students at ETH is increasing year after year, and with it the correction workload, more and more multiple choice questions are being used in examinations to prevent the correction workload from becoming immeasurable. While a lot of time is undeniably saved during correction, the question arises as to whether an exam with single choice questions is just as meaningful as an exam with open-ended questions. In other words, the question arises as to how single choice questions should be designed so that they can capture the students' level of performance as well as possible. As the worked out solution is not recorded in single choice questions, it is not possible to understand the students' thoughts. On the one hand, this makes it difficult to recognize a poorly constructed question. On the other hand, it is all the more important that single choice questions are well thought out so that the examination result is meaningful. There is already a lot of literature on this, e.g. Haladyna (2004), Abramovitz et al. (2005) and Krebs (2019) that also provide examples showing how not to formulate single choice questions. For example, when creating these, care should be taken to ensure that the question is clearly formulated and, in particular, that there are no double negatives. All answer options should also be of a similar length and Hembree (1987) suggests that a number of 3 distractors is best – but this is only backed by references to previous theoretical work, since there were not enough relevant studies in their meta-analysis. Among other things, the literature recommends the use of functioning distractors. This means that incorrect answer options are available for selection, which are also chosen by a certain percentage of students. Faulkner (1977) adds that suitable alternative answers can be very difficult indeed to find. There are several studies investigating whether distractors work and describing how to find working distractors, see Tarrant, Ware and Mohammed (2009) and Ali, Carr and Ruit (2016).
While these two studies investigated medical questions where the answers were terms, in this paper we investigate mathematics questions with numerical answers. Lerchenberger and Donner (2024) study mathematical single choice tasks and state that it seems of utmost importance that task designers should be aware of the fact that the choice of distractors has

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an oversized influence on the average score and is therefore of great importance. Hence, it requires special attention. They define the concept of a trap and subdivide this in various categories and mention that the inclusion of potential traps can be used as a conscious means to create tricky tasks that require self-monitoring of the students in an exam setting.

Feng et al. (2024) explains as the name implies, distractors in single choice questions are typically formulated to align with the common errors students would make or misconceptions students would exhibit. These distractors are chosen because students either i) lack the necessary knowledge of the skills tested in the question to accurately identify the key as the correct answer, or ii) hold misconceptions that result in selecting a specific distractor as the correct answer. While single choice questions offer many advantages for student knowledge evaluation, manually crafting high-quality questions is a demanding and labour-intensive process. Specifically, high-quality distractors should be plausible enough to mislead students and not so evidently incorrect to be identified easily. Furthermore, we investigate to what extent the distractors influence how confident students are about the correctness of their answer, as well as how much time they need to answer the questions. As far as we are aware, the effect on confidence and time has not yet been investigated in any study.

Methodology

In order to investigate the effect of different distractors, we conducted an experiment with the students of a large calculus lecture course for the department D-BAUG (civil and environmental engineers) at the ETH Zurich. All students who were enrolled in this course were allowed to participate. The experiment was conducted as a single choice quiz via Moodle, the teaching and learning platform of ETH Zurich. The quiz consisted of 10 single choice questions on integral calculus, each with 4 possible answers, and related to the previously covered lecture content. For each question, exactly one of the 4 possible answers was correct.
When formulating the questions, particular attention was paid to the following points, which according to the literature should always be taken into account with multiple or single choice questions (cf. Haladyna (2004), Krebs (2019)):

- The questions are linguistically clear and formulated as briefly as possible.
- The questions do not contain any superfluous information.
- No personal names are used.
- There is only one correct answer. The question does not allow for different interpretations.
- All possible answers are visually comparable.

For each of the 10 mathematical questions, the students also had to indicate how confident they were about their answer (very confident - somewhat confident - somewhat uncertain - very uncertain). While all students were asked the same questions in the same order, the three distractors, i.e. the incorrect answer options, were different in each case. The students were randomly divided into two groups, with one set of distractors used for each group. (More details on the choice of distractors in the next paragraph).

After the mathematical questions, we added another question asking about the total time needed, where the students could choose from 5 options (0-15 min, 16-30 min, 31-45 min, 46-60 min, >60 min).

The students solved the quiz without supervision in their free time. As an incentive for participating, the students then received a ‘bonus point’ (independent of their performance), which indirectly led to a grade bonus of around 0.015 on the final exam (on a scale from 1 to 6). Due to the experimental design, it was to be expected that some students would use unauthorised aids or would answer the questions at random with as little time as possible. In order to exclude these students from the analysis, the quiz included an additional question about whether they had answered the questions conscientiously. This was accompanied by a note that the answer to this question had no influence on the bonus points awarded.

The quiz was completed by a total of 170 students out of approximately 280, with 79 in the first group and 91 in the second group. When cleaning the data, those who did not tick ‘yes’ to the question on conscientiousness or who did not agree to their answers being analysed in anonymised form were excluded. As a result, a total of 18 students were excluded. We also excluded four other students who had only scored one or two points and were therefore below the expected value of 2.5 points. As a result, the data of 72 subjects in Group 1 and 76 subjects in Group 2 were analysed.

For the evaluation of the data, we opted for a mixed form: On the one hand, we draw some very obvious and interesting conclusions just by ‘looking’ at the data. On the other hand, we analyse the data with a mixed binary regression model. In particular, this method provides a significant statement about the entire experiment, while direct observations relate more to individual tasks.

Choice of distractors

For Group 1 we followed the literature and tried to design functioning distractors, i.e. distractors that will actually be chosen by a certain percentage of the students. Our guiding principle here was to detect common errors and misconceptions and build distractors from them. The term ‘common’ here, of course, refers only to the errors we predicted (from experience and analysing old exams). While some of these distractors actually turned out to be enormously attractive during the analysis, others were hardly ever chosen.

For Group 2 we constructed three distractors, which are visibly similar to the ones of Group 1, but which are not obtained whilst making the errors used for Group 1. ‘Visibly similar’ here means that there are no distractors that are out of the ordinary, which would be immediately excluded even without being able to solve the task. Care was therefore taken to ensure that the same type of distractors (natural number, rational number, trigonometric expression, expression with π, expression with e, roots) as in Group 1 were also offered as possible answers in Group 2. The order of magnitude of the distractors in Group 2 was also comparable to that in Group 1.

In general, we followed the principle that it should be hard to deduce the right answer by simply looking at all answers and using symmetry arguments (cf. Question 1 below).

In the following section we have picked two questions that demonstrate particularly well how we constructed the distractors for both groups - the total set of questions together with an explanation of the distractors in Group 1 can be found in the appendix.

Examples

Question 1 is a very basic exercise about integration by parts. The correct solution is

$$\int_0^1 2x \cdot \mathrm{e}^x \,\mathrm{d}x = [2x\mathrm{e}^x]_0^1 - \int_0^1 2\mathrm{e}^x \,\mathrm{d}x = 2\mathrm{e} - (2\mathrm{e} - 2) = 2.$$

Question 1

Calculate the following integral: $\int_0^1 2x\mathrm{e}^x \,\mathrm{d}x =$

Answer options Group 1:

(a) 0

✓ (b) 2

(c) e

(d) $4\mathrm{e}-2$

Answer options Group 2:

(a) 1

✓ (b) 2

(c) $3\mathrm{e}$

(d) $5\mathrm{e}+1$

Figure 1: Question 1 from the quiz.

The following common errors were used to design the distractors:

(a) If a student accidentally differentiates in the first summand as well, he would get $[2\mathrm{e}^x]_0^1 - \int_0^1 2\mathrm{e}^x \,\mathrm{d}x = [2\mathrm{e}^x]_0^1 - [2\mathrm{e}^x]_0^1 = 0$. The same distractor can also be obtained with another error, namely by assuming $\mathrm{e}^0 = 0$ at the very end of the correct calculation. Here, we would like to point out that the answer 0 can already be identified as incorrect purely geometrically, since a non-negative function is being integrated. Nevertheless, we decided to offer this distractor as a choice, because students often calculate stubbornly without questioning the result.

(c) If instead of the product both factors are integrated individually, one gets $[x^2\mathrm{e}^x]_0^1 = \mathrm{e}$.

(d) If the minus sign is forgotten in the process of integrating by parts, one gets $[2x\mathrm{e}^x]_0^1 + \int_0^1 2\mathrm{e}^x \,\mathrm{d}x = 2\mathrm{e} + (2\mathrm{e}-2) = 4\mathrm{e}-2$.

Care was taken to ensure that the distractors in Group 2 were visibly similar to those in Group 1. Instead of 0 we offered 1 as the first distractor, since they both are very special integers. Instead of e, the answer 3e was offered, as this also contains Euler's number and does not appear more complicated. In accordance with the distractor $4\mathrm{e}-2$ from Group 1, we built a linear combination of Euler's number and the number 1, namely $5\mathrm{e}+1$, as the last distractor. Note that in both cases two answers were integers and two linear combinations with e. Hereby we hoped not to focus too much on one or the other as the symmetry does not give away anything about the nature of the answer (integer or irrational).

Question 8 deals with a triple integral that needs to be solved with the 'change of variables' method. The correct answer can be calculated using cylindrical coordinates resulting in

$$\iiint_Z \frac{x^2+y^2}{z^2}\,\mathrm{d}V = \int_1^3 \int_0^{2\pi} \int_0^2 \frac{r^2}{z^2} r \,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z = 2\pi \cdot \left[-\frac{1}{z}\right]_1^3 \cdot \left[\frac{1}{4}r^4\right]_0^2 = 2\pi \cdot \frac{2}{3} \cdot 4 = \frac{16\pi}{3}.$$

Question 8

Consider the cylinder

$$Z = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x^2 + y^2 \leq 4 \ , \ 1 \leq z \leq 3\}$$

Calculate the following integral: $\iiint_Z \frac{x^2 + y^2}{z^2} \, \mathrm{d}V =$

Answer options Group 1:

(a) $\frac{32\pi}{9}$
✓ (b) $\frac{16\pi}{3}$
(c) $\frac{32\pi}{3}$
(d) $\frac{256\pi}{3}$

Answer options Group 2:

(a) $\frac{24\pi}{9}$
✓ (b) $\frac{16\pi}{3}$
(c) $\frac{48\pi}{3}$
(d) $\frac{140\pi}{3}$

Figure 2: Question 8 from the quiz.

The following distractors, based on common errors, were selected for Group 1:

(a) If the r in the volume element is forgotten, one gets
$\int_1^3 \int_0^{2\pi} \int_0^2 \frac{r^2}{z^2} \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z = 2\pi \cdot \left[-\frac{1}{z}\right]_1^3 \cdot \left[\frac{1}{3} r^3\right]_0^2 = 2\pi \cdot \frac{2}{3} \cdot \frac{8}{3} = \frac{32\pi}{9}$.

(c) If both the r is forgotten in the volume element and r is used for $x^2 + y^2$ and hence, the upper limit of r is taken as 4, one gets
$\int_1^3 \int_0^{2\pi} \int_0^4 \frac{r}{z^2} \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z = 2\pi \cdot \left[-\frac{1}{z}\right]_1^3 \cdot \left[\frac{1}{2} r^2\right]_0^4 = 2\pi \cdot \frac{2}{3} \cdot 8 = \frac{32\pi}{3}$.

(d) If the upper limit of the radius is taken as 4 instead of 2, the resulting calculation is
$\int_1^3 \int_0^{2\pi} \int_0^4 \frac{r^2}{z^2} \, r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z = 2\pi \cdot \left[-\frac{1}{z}\right]_1^3 \cdot \left[\frac{1}{4} r^4\right]_0^4 = 2\pi \cdot \frac{2}{3} \cdot 64 = \frac{256\pi}{3}$.

Again, we have chosen visibly very similar numbers as distractors for Group 2.

Analysis

First, we examined the extent to which the total number of points achieved (1 point per correctly answered question) of students in Group 1 differed from those in Group 2. Since the students were randomly divided into the two groups, it can be assumed that the total number of points achieved in both groups is normally distributed and has a similar variance. Thus, a Student t-test can be carried out, whereby the null hypothesis is that there is no difference between the two groups in the total scores achieved by the students. The mean received points were 0.4 higher in Group 2 than in Group 1 (p = 0.019).

In the next step we dived deeper into the details and analysed how well the groups answered each question. In contrast to the total score, the data set here is binary. By simply screening the data much can be observed when studying the following issues:

- Differences between the two groups with regard to the correctness of the answer
- Frequency with which the respective distractors were chosen
- Students' confidence about the correctness of their answer depending on the group

We will show our findings in an example. Analysing the answer behaviour in Question 8 (Table 1, Table 2), we observe that Group 2 outperforms Group 1. We observe however that in Group 1 not only more often the wrong distractors were chosen, but that this also happens with greater confidence. In Group 2 it seems that wrong answers were mainly picked by guessing.

Group 1	Total	Very confident	Somewhat confident	Somewhat uncertain	Very uncertain
(a)	17	2	7	5	3
(b)	29	13	13	3	0
(c)	13	2	4	2	5
(d)	12	5	4	0	3

Table 1: Answers to Question 8 by Group 1.

Group 2	Total	Very confident	Somewhat confident	Somewhat uncertain	Very uncertain
(a)	9	2	1	2	4
(b)	47	15	14	9	9
(c)	11	0	0	3	8
(d)	6	0	0	2	4

Table 2: Answers to Question 8 by Group 2.

For the deeper statistical analysis, we used a mixed binary regression. Correctness was modelled as the dependent variable, and group membership and certainty, as well as their interaction, were the fixed predictors. In addition, both the corresponding student ID and the question ID were included as random intercepts. The random intercepts take into account the dependency between individual questions and individual students. It is therefore assumed that the correctness of the answer depends on which question it is (difficulty of the question) and which student has answered it (mathematical ability of the student).

We would like to specifically mention the following significant results, which support the observations made in the descriptive evaluation:

1) Ignoring security levels, students from Group 2 perform better overall. (Group 1: 65 % correct answers, Group 2: 81 % correct answers). This difference is statistically significant (OR: 0.43, $p<0.001$).
2) While the very or somewhat uncertain students in both groups perform similarly poorly, there are significant differences between the probabilities for a correct answer between the two groups for the somewhat confident and very confident students, with Group 2 performing significantly better (Table 3, Figure 3).

Certainty	Odds Ratio	SE	Z-Ratio	P-Value
Very Uncertain	0.58	0.20	-1.59	0.112
Somewhat Uncertain	1.13	0.34	0.41	0.683
Somewhat Confident	0.22	0.07	-4.46	<0.0001
Very Confident	0.23	0.09	-3.84	0.0001

Table 3: Posthoc Contrasts between Group 1 and Group 2 for each level of certainty. P-values are uncorrected.

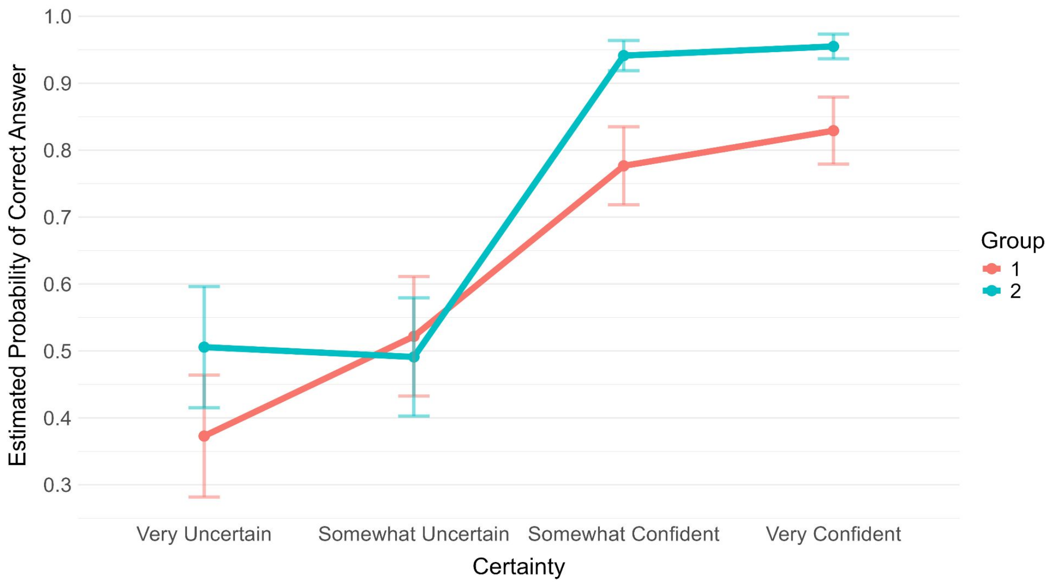

Figure 3: Interaction between Group Membership and Certainty on the Estimated Probability of Correct Responses.

Summary and outlook

As we had expected, the students from Group 2 performed significantly better overall. The obvious explanation is as follows: When students from Group 2 make a common error or have a misconception that we used to create the distractors of Group 1, they get a result that is not available for selection. This means that they have to rethink their answer and thus have the chance to still get the correct solution after all. However, the students from Group 1 which are making the same mistake get a result that corresponds to one of the answer options. Consequently, they mark this incorrect answer and move on to the next question.

It is also explainable that students from Group 1 are excessively often somewhat or even very confident compared to those from Group 2, although they picked a distractor: Since these students arrive at a result from a common error or misconception that is offered for selection, they feel confirmed in their (wrong) answer. They are therefore lulled into a false sense of security by these distractors.

However, one hypothesis that was not confirmed by the experiment is the following: Since the students from Group 2 receive a result that is not available for selection if they calculate incorrectly and therefore have to reconsider their calculation, we expected that they would need more time overall to answer the questions. Yet, it appears that both groups needed roughly the same amount of time to answer the questions. On the one hand, however, this corresponding question only gave us an imprecise time indication, and on the other hand we have no data on the time taken per task. Although we do not see any noticeable differences between the groups in terms of the time required, we cannot rule out the possibility that there are some.

We observed - as was to be expected - that for very hard (only few students could answer correctly) or for very easy (almost all students could answer correctly) questions, the role of the distractors is not so important. However, for medium difficulty questions the distractors play a crucial role in the answering behaviour of the students. It is important to be aware of this when creating multiple or single choice tasks. In our opinion, there is no one right type of distractor. Distractors like those in Group 2 give students the opportunity to realise their mistakes themselves and learn directly from them. This may be very good in practising situations, as they do not give the students a false sense of security and instead gives an opportunity to learn from their own mistakes. For graded tests on the other hand, it may be better to use distractors of the type in Group 1. Special care is required if students receive different versions for an exam. In this case, it is absolutely essential that the distractors in all versions are comparable to each other. In particular, we advise never to have one version with distractors as in Group 1 and another version with distractors as in Group 2. In this case, students' exam success would strongly depend on which version of the exam they receive.

Moreover, when creating distractors like those of Group 1, one needs to be very careful: It is important to have a clear idea about what is tested in the question and what common errors and misconceptions could look like. Common errors that could occur, but are not actually due to the topic at hand in this question, should be avoided e.g. when testing integration by parts the minus in the formula seems crucial. However, by adding for example trigonometric functions, other sign errors could occur resulting perhaps even in the correct answer by doing two things wrong.

As we were surprised that the students in Group 2 did not need more time than the ones in Group 1 and as we cannot exclude that this was because of imprecise measurement, we suggest for future research to do a more careful analysis of the time used when different types of distractors are chosen.

As our study only spanned a short time period, we were not able to say anything about the differences between the two groups in the long-term learning. We think it would be very interesting to study this more carefully and to find out whether e.g. the Group 2 type of distractors led to better and deeper understanding.

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Appendix: Questions from Moodle Quiz with explanation of choice of distractors

Question 1

Calculate the following integral: $\int_0^1 2xe^x \, dx =$

Answer options Group 1:	Answer options Group 2:
(a) 0	(a) 1
✓ (b) 2	✓ (b) 2
(c) e	(c) $3e$
(d) $4e - 2$	(d) $5e + 1$

Correct calculation: $\int_0^1 2x \cdot e^x \, dx = [2xe^x]_0^1 - \int_0^1 2e^x \, dx = 2e - (2e - 2) = 2$

Choice of distractors for Group 1:

(a) If a student accidentally differentiates in the first summand as well, he would get $[2e^x]_0^1 - \int_0^1 2e^x \, dx = [2e^x]_0^1 - [2e^x]_0^1 = 0$. The same distractor can also be obtained with another error, namely by assuming $e^0 = 0$ at the very end of the correct calculation. Here, we would like to point out that the answer 0 can already be identified as incorrect purely geometrically, since a non-negative function is being integrated. Nevertheless, we decided to offer this distractor as a choice, because students often calculate stubbornly without questioning the result.

(c) If instead of the product both factors are integrated individually, one gets $[x^2 e^x]_0^1 = e$.

(d) If the minus sign is forgotten in the process of integrating by parts, one gets $[2xe^x]_0^1 + \int_0^1 2e^x \, dx = 2e + (2e - 2) = 4e - 2$.

Question 2

Use the substitution $x = \sin(u)$ to calculate the following integral: $\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x =$

Answer options Group 1:

✓ (a) $\frac{\pi}{4}$
(b) 1
(c) $\sin(1)$
(d) $\frac{1}{2} + \frac{1}{4}\sin(2)$

Answer options Group 2:

✓ (a) $\frac{\pi}{4}$
(b) $\frac{\pi}{3}$
(c) 2
(d) $\frac{1}{4} + \sin(2)$

Correct calculation:

$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \int_0^{\pi/2} \sqrt{1-\sin(u)^2}\cos(u)\,\mathrm{d}u = \int_0^{\pi/2} \cos(u)^2\,\mathrm{d}u = \frac{1}{2}\int_0^{\pi/2}(1+\cos(2u))\,\mathrm{d}u$
$= \frac{1}{2}\left[u + \frac{1}{2}\sin(2u)\right]_0^{\pi/2} = \frac{\pi}{4}$

Choice of distractors for Group 1:

(b) If instead of $\cos(u)\,\mathrm{d}u$ one writes only $\mathrm{d}u$ after the substitution, one gets $\int_0^{\pi/2} \cos(u)\,\mathrm{d}u = [\sin(u)]_0^{\pi/2} = 1.$
(d) If the integral limits are not adjusted when making the substitution, one gets $\int_0^1 \cos(u)^2\,\mathrm{d}u = \frac{1}{2}\left[u + \frac{1}{2}\sin(2u)\right]_0^1 = \frac{1}{2} + \frac{1}{4}\sin(2).$
(c) If both mistakes are done at the same time, one gets $\int_0^1 \cos(u)\,\mathrm{d}u = [\sin(u)]_0^1 = \sin(1).$

Question 3

Calculate the following integral: $\int_0^2 \int_0^3 \left(3x^2 + 2y\right)\,\mathrm{d}x\,\mathrm{d}y =$

Answer options Group 1:

(a) 17
(b) 31
(c) 42
✓ (d) 66

Answer options Group 2:

(a) 15
(b) 34
(c) 53
✓ (d) 66

Correct calculation:

$\int_0^2 \int_0^3 (3x^2 + 2y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 [x^3 + 2xy]_0^3\,\mathrm{d}y = \int_0^2 (27 + 6y)\,\mathrm{d}y = [27y + 3y^2]_0^2 = 54 + 12 = 66$

Choice of distractors for Group 1:

(b) If the plus gets handled incorrectly and instead two single integrals are calculated, one gets $\int_0^2 2y\,\mathrm{d}y + \int_0^3 3x^2\,\mathrm{d}x = [y^2]_0^2 + [x^3]_0^3 = 4 + 27 = 31.$

(a) If additionally to the already described mistake, also the limits for the integrals are interchanged, one gets $\int_0^2 3x^2\,\mathrm{d}x + \int_0^3 2y\,\mathrm{d}y = [x^3]_0^2 + [y^2]_0^3 = 8 + 9 = 17$.

(c) If the double integral is calculated with limits for x and y interchanged, one gets $\int_0^3 \int_0^2 (3x^2 + 2y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^3 [x^3 + 2xy]_0^2\,\mathrm{d}y = \int_0^3 (8 + 4y)\,\mathrm{d}y = [8y + 2y^2]_0^3 = 42$.

Question 4

If $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, then $\int_{-2}^{2} \int_{(x+2)^2}^{8\sqrt{x+2}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x =$

Answer options Group 1:

(a) $\int_{(x+2)^2}^{8\sqrt{x+2}} \int_{-2}^{2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

(b) $\int_{0}^{16} \int_{-2}^{2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

✓ (c) $\int_{0}^{16} \int_{\frac{y^2}{64}-2}^{\sqrt{y}-2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

(d) $\int_{0}^{16} \int_{\sqrt{y}-2}^{\frac{y^2}{64}-2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

Answer options Group 2:

(a) $\int_{(x+2)^2}^{8\sqrt{x+2}} \int_{0}^{4} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

(b) $\int_{0}^{4} \int_{-2}^{2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

✓ (c) $\int_{0}^{16} \int_{\frac{y^2}{64}-2}^{\sqrt{y}-2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

(d) $\int_{0}^{16} \int_{\sqrt{y}}^{y^2-2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

Correct calculation: Since $(0 \leq)\ y \leq 8\sqrt{x+2}$ if and only if $\frac{y^2}{64} - 2 \leq x$, and $(x+2)^2 \leq y$ if and only if $x \leq \sqrt{y} - 2$, the correct answer is (c).

Choice of distractors for Group 1:

(a) This is obtained by interchanging the order of integration without adjusting the limits.
(b) This is obtained by adjusting the limits for the y-Integral only.
(d) This is obtained if the upper and lower limits of x are swapped.

Question 5

Calculate the following integral: $\int_0^3 \int_0^2 \int_{-1}^{0} x^2 y\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z =$

Answer options Group 1:

(a) $\frac{2}{3}$

✓ (b) 2

(c) 6

(d) 18

Answer options Group 2:

(a) $\frac{2}{5}$

✓ (b) 2

(c) 4

(d) 15

Correct calculation:
$\int_0^3 \int_0^2 \int_{-1}^0 x^2 y \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int_0^3 1 \, \mathrm{d}z \cdot \int_0^2 y \, \mathrm{d}y \cdot \int_{-1}^0 x^2 \, \mathrm{d}x = [z]_0^3 \cdot \left[\frac{1}{2}y^2\right]_0^2 \cdot \left[\frac{1}{3}x^3\right]_{-1}^0 = 3 \cdot 2 \cdot \frac{1}{3} = 2.$

Choice of distractors for Group 1:

(a) This is obtained if the integration by z is completely ignored.
(c) This is obtained if the integrand is ignored and simply the volume of the cuboid $[0,3] \times [0,2] \times [-1,0]$ is calculated.
(d) If the limits of integration for x and z are interchanged, one gets $\int_0^3 x^2 \, \mathrm{d}x \cdot \int_0^2 y \, \mathrm{d}y \cdot \int_{-1}^0 1 \, \mathrm{d}z = \left[\frac{1}{3}x^3\right]_0^3 \cdot \left[\frac{1}{2}y^2\right]_0^2 \cdot [z]_{-1}^0 = 9 \cdot 2 \cdot 1 = 18.$

Question 6

The area in the I. quadrant which is bounded by the curve $y = 5 - x^2$ as well as the x- and y-axis, gets rotated around the **y-axis**. What is the volume of the resulting body?

Answer options Group 1:

✓ (a) $\frac{25}{2}\pi$
(b) $\frac{1000}{3}\pi$
(c) $\frac{10\sqrt{5}}{3}\pi$
(d) $\frac{40\sqrt{5}}{3}\pi$

Answer options Group 2:

✓ (a) $\frac{25}{2}\pi$
(b) $\frac{800}{3}\pi$
(c) $\frac{5\sqrt{7}}{3}\pi$
(d) $\frac{30\sqrt{7}}{3}\pi$

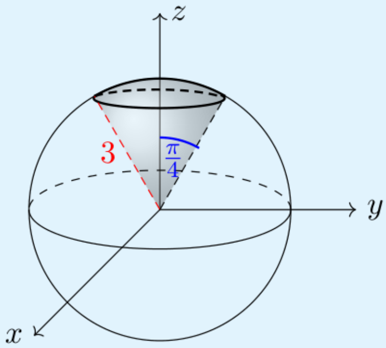
Correct calculation: $V = \pi \cdot \int_0^5 (\sqrt{5-y})^2 \, \mathrm{d}y = \pi \cdot \int_0^5 (5-y) \, \mathrm{d}y = \pi \cdot \left[5y - \frac{1}{2}y^2\right]_0^5 = \frac{25}{2}\pi.$

Choice of distractors for Group 1:

(b) With rotation around the x-axis and hence, no use of the inverse function, one gets
$\pi \cdot \int_0^5 (5-x^2)^2 \, \mathrm{d}x = \pi \cdot \int_0^5 (25 - 10x^2 + x^4) \, \mathrm{d}x = \pi \cdot \left[25x - \frac{10}{3}x^3 + \frac{1}{5}x^5\right]_0^5 = \frac{1000}{3}\pi.$
(d) The same mistake as before with the upper limit taken as $\sqrt{5}$ gives
$\pi \cdot \int_0^{\sqrt{5}} (5-x^2)^2 \, \mathrm{d}x = \pi \cdot \int_0^{\sqrt{5}} (25 - 10x^2 + x^4) \, \mathrm{d}x = \pi \cdot \left[25x - \frac{10}{3}x^3 + \frac{1}{5}x^5\right]_0^{\sqrt{5}} = \frac{40\sqrt{5}}{3}\pi.$
(c) If the mistake is to forget to square the integrand, one gets
$\pi \cdot \int_0^5 \sqrt{5-y} \, \mathrm{d}y = \pi \cdot \left[-\frac{2}{3}(5-y)^{3/2}\right]_0^5 = \frac{10\sqrt{5}}{3}\pi.$

Question 7

Calculate the volume of the following spherical cut-out:



Answer options Group 1:

(a) $V = 9\pi$

(b) $V = 9\sqrt{2}\pi$

(c) $V = 9\left(1 - \frac{1}{\sqrt{2}}\right)\pi$

✓ (d) $V = 9\left(2 - \sqrt{2}\right)\pi$

Answer options Group 2:

(a) $V = 8\pi$

(b) $V = 3\sqrt{2}\pi$

(c) $V = 6\left(1 + \sqrt{2}\right)\pi$

✓ (d) $V = 9\left(2 - \sqrt{2}\right)\pi$

Correct calculation:

$V = \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 1 \cdot r^2 \sin(\vartheta)\, \mathrm{d}r\, \mathrm{d}\vartheta\, \mathrm{d}\varphi = 2\pi \cdot \int_0^3 r^2\, dr \cdot \int_0^{\pi/4} \sin(\vartheta)\, \mathrm{d}\vartheta$
$= 2\pi \cdot \left[\frac{1}{3}r^3\right]_0^3 \cdot [-\cos(\vartheta)]_0^{\pi/4} = 2\pi \cdot 9 \cdot \left(-\frac{\sqrt{2}}{2} + 1\right) = 9(2 - \sqrt{2})\pi.$

Choice of distractors for Group 1:

(a) Since with angle π, one would get the volume of the whole sphere, a possible mistake is to think that with angle $\pi/4$ one gets a fourth of it, i.e. $\frac{1}{4} \cdot \frac{4\pi \cdot 3^3}{3} = 9\pi$.

(b) Taking the wrong volume element $r^2\cos(\vartheta)\, \mathrm{d}r\, \mathrm{d}\vartheta\, \mathrm{d}\varphi$, one gets
$\int_0^{2\pi} \int_0^{\pi/4} \int_0^3 1 \cdot r^2 \cos(\vartheta)\, \mathrm{d}r\, \mathrm{d}\vartheta\, \mathrm{d}\varphi = 2\pi \cdot \int_0^3 r^2\, \mathrm{d}r \cdot \int_0^{\pi/4} \cos(\vartheta)\, \mathrm{d}\vartheta = 2\pi \cdot \left[\frac{1}{3}r^3\right]_0^3 \cdot [\sin(\vartheta)]_0^{\pi/4}$
$= 2\pi \cdot 9 \cdot \frac{\sqrt{2}}{2} = 9\sqrt{2}\pi.$

(c) Taking the wrong volume element $r\sin(\vartheta)\, \mathrm{d}r\, \mathrm{d}\vartheta\, \mathrm{d}\varphi$, one gets
$\int_0^{2\pi} \int_0^{\pi/4} \int_0^3 1 \cdot r \sin(\vartheta)\, \mathrm{d}r\, \mathrm{d}\vartheta\, \mathrm{d}\varphi = 2\pi \cdot \int_0^3 r\, \mathrm{d}r \cdot \int_0^{\pi/4} \sin(\vartheta)\, \mathrm{d}\vartheta = 2\pi \cdot \left[\frac{1}{2}r^2\right]_0^3 \cdot [-\cos(\vartheta)]_0^{\pi/4}$
$= 2\pi \cdot \frac{9}{2} \cdot \left(-\frac{\sqrt{2}}{2} + 1\right) = 9\left(1 - \frac{1}{\sqrt{2}}\right)\pi.$

Other potential mistakes in this exercise could have been to completely forget the volume element or to take the wrong volume element r as for polar coordinates. However, these mistakes would leave to results with π^2, which looks somewhat different and hence, we didn't provide them as distractors.

Question 8

Consider the cylinder

$$Z = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x^2 + y^2 \leq 4\ ,\ 1 \leq z \leq 3\}$$

Calculate the following integral: $\iiint_Z \frac{x^2+y^2}{z^2}\,\mathrm{d}V =$

Answer options Group 1:

- (a) $\frac{32\pi}{9}$
- ✓ (b) $\frac{16\pi}{3}$
- (c) $\frac{32\pi}{3}$
- (d) $\frac{256\pi}{3}$

Answer options Group 2:

- (a) $\frac{24\pi}{9}$
- ✓ (b) $\frac{16\pi}{3}$
- (c) $\frac{48\pi}{3}$
- (d) $\frac{140\pi}{3}$

Correct calculation: Using cylindrical coordinates, one gets

$$\iiint_Z \frac{x^2+y^2}{z^2}\,\mathrm{d}V = \int_1^3\int_0^{2\pi}\int_0^2 \frac{r^2}{z^2} r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z = \int_0^{2\pi} 1\,\mathrm{d}\varphi \cdot \int_1^3 \frac{1}{z^2}\,\mathrm{d}z \cdot \int_0^2 r^3\,\mathrm{d}r = 2\pi \cdot \left[-\frac{1}{z}\right]_1^3 \cdot \left[\frac{1}{4}r^4\right]_0^2$$
$$= 2\pi \cdot \frac{2}{3} \cdot 4 = \frac{16\pi}{3}.$$

Choice of distractors for Group 1:

(a) If r is forgotten in the volume element, one gets
$$\int_1^3\int_0^{2\pi}\int_0^2 \frac{r^2}{z^2}\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z = \int_0^{2\pi} 1\,\mathrm{d}\varphi \cdot \int_1^3 \frac{1}{z^2}\,\mathrm{d}z \cdot \int_0^2 r^2\,\mathrm{d}r = 2\pi \cdot \left[-\frac{1}{z}\right]_1^3 \cdot \left[\frac{1}{3}r^3\right]_0^2 = 2\pi \cdot \frac{2}{3} \cdot \frac{8}{3} = \frac{32\pi}{9}.$$

(d) If the upper limit of the radius is wrongly taken as 4 instead of 2, one gets
$$\int_1^3\int_0^{2\pi}\int_0^4 \frac{r^2}{z^2} r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z = \int_0^{2\pi} 1\,\mathrm{d}\varphi \cdot \int_1^3 \frac{1}{z^2}\,\mathrm{d}z \cdot \int_0^4 r^3\,\mathrm{d}r = 2\pi \cdot \left[-\frac{1}{z}\right]_1^3 \cdot \left[\frac{1}{4}r^4\right]_0^4 = 2\pi \cdot \frac{2}{3} \cdot 64$$
$$= \frac{256\pi}{3}.$$

(c) If at the same time the r is forgotten in the volume element and $x^2 + y^2$ is wrongly taken as r, one gets
$$\int_1^3\int_0^{2\pi}\int_0^4 \frac{r}{z^2}\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z = \int_0^{2\pi} 1\,\mathrm{d}\varphi \cdot \int_1^3 \frac{1}{z^2}\,\mathrm{d}z \cdot \int_0^4 r\,\mathrm{d}r = 2\pi \cdot \left[-\frac{1}{z}\right]_1^3 \cdot \left[\frac{1}{2}r^2\right]_0^4 = 2\pi \cdot \frac{2}{3} \cdot 8 = \frac{32\pi}{3}.$$

Question 9

We consider the circular ring

$$R = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 3\}$$

Calculate the following integral: $\iint_R \frac{y^2}{x^2+y^2}\, \mathrm{d}A =$

Answer options Group 1:

✓ (a) π
(b) $\frac{3\pi}{2}$
(c) 4π
(d) $\left(\sqrt{3}-1\right)\pi$

Answer options Group 2:

✓ (a) π
(b) $\frac{5\pi}{2}$
(c) 3π
(d) $\left(2-\sqrt{3}\right)\pi$

Correct calculation: Using polar coordinates one gets

$$\iint_R \frac{y^2}{x^2+y^2}\, \mathrm{d}A = \int_0^{2\pi}\int_1^{\sqrt{3}} \frac{r^2\sin^2(\varphi)}{r^2} r\, \mathrm{d}r\, \mathrm{d}\varphi = \int_0^{2\pi} \sin^2(\varphi)\, \mathrm{d}\varphi \cdot \int_1^{\sqrt{3}} r\, \mathrm{d}r$$
$$= \int_0^{2\pi} \frac{1}{2}(1-\cos(2\varphi))\mathrm{d}\varphi \cdot \int_1^{\sqrt{3}} r\, \mathrm{d}r = \left[\frac{1}{2}\left(\varphi - \frac{1}{2}\sin(2\varphi)\right)\right]_0^{2\pi} \cdot \left[\frac{1}{2}r^2\right]_1^{\sqrt{3}} = \pi \cdot 1 = \pi.$$

Choice of distractors for Group 1:

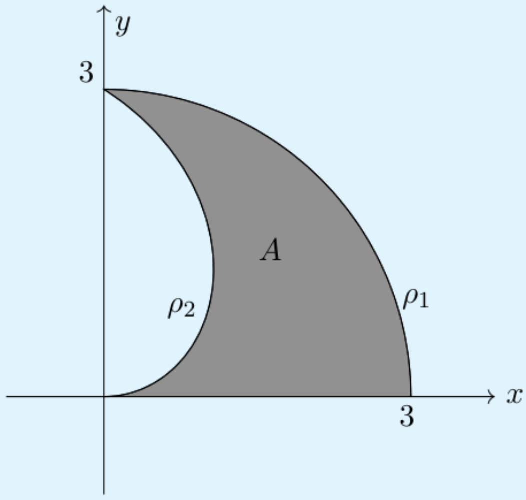
(b) If the lower limit of the radius is mistakenly taken as 0, one gets
$$\int_0^{2\pi}\int_0^{\sqrt{3}} \frac{r^2\sin^2(\varphi)}{r^2} r\, \mathrm{d}r\, \mathrm{d}\varphi = \int_0^{2\pi} \sin^2(\varphi)\, \mathrm{d}\varphi \cdot \int_0^{\sqrt{3}} r\, \mathrm{d}r = \left[\frac{1}{2}\left(\varphi - \frac{1}{2}\sin(2\varphi)\right)\right]_0^{2\pi} \cdot \left[\frac{1}{2}r^2\right]_0^{\sqrt{3}}$$
$$= \pi \cdot \frac{3}{2} = \frac{3\pi}{2}.$$

(c) If the upper limit of the radius is mistakenly taken as 3 instead of $\sqrt{3}$, one gets
$$\int_0^{2\pi}\int_1^{3} \frac{r^2\sin^2(\varphi)}{r^2} r\, \mathrm{d}r\, \mathrm{d}\varphi = \int_0^{2\pi} \sin^2(\varphi)\, \mathrm{d}\varphi \cdot \int_1^{3} r\, \mathrm{d}r = \left[\frac{1}{2}\left(\varphi - \frac{1}{2}\sin(2\varphi)\right)\right]_0^{2\pi} \cdot \left[\frac{1}{2}r^2\right]_1^{3}$$
$$= \pi \cdot \left(\frac{9}{2} - \frac{1}{2}\right) = 4\pi.$$

(d) If the factor r is forgotten in the area element, one gets
$$\int_0^{2\pi}\int_1^{\sqrt{3}} \frac{r^2\sin^2(\varphi)}{r^2}\, \mathrm{d}r\, \mathrm{d}\varphi = \int_0^{2\pi} \sin^2(\varphi)\, \mathrm{d}\varphi \cdot \int_1^{\sqrt{3}} 1\, \mathrm{d}r = \pi \cdot \left(\sqrt{3}-1\right) = \left(\sqrt{3}-1\right)\pi.$$

Question 10

Calculate the area in the first quadrant which gets enclosed by the curves with polar coordinate representation $\rho_1(\varphi) = 3$ and $\rho_2(\varphi) = \frac{6\varphi}{\pi}$.



Answer options Group 1:

(a) $A = \frac{9\pi}{8}$
✓ (b) $A = \frac{3\pi}{2}$
(c) $A = \frac{9\pi}{4}$
(d) $A = \frac{57\pi}{4}$

Answer options Group 2:

(a) $A = \frac{5\pi}{4}$
✓ (b) $A = \frac{3\pi}{2}$
(c) $A = \frac{17\pi}{8}$
(d) $A = \frac{45\pi}{4}$

Correct calculation:

$$A = \frac{1}{2}\int_0^{\pi/2} \rho_1(\varphi)^2 \,\mathrm{d}\varphi - \frac{1}{2}\int_0^{\pi/2} \rho_2(\varphi)^2 \,\mathrm{d}\varphi = \frac{1}{2}\int_0^{\pi/2} 3^2 \,\mathrm{d}\varphi - \frac{1}{2}\int_0^{\pi/2} \left(\frac{6\varphi}{\pi}\right)^2 \mathrm{d}\varphi = \frac{9\pi}{4} - \frac{18}{\pi^2}\left[\frac{1}{3}\varphi^3\right]_0^{\pi/2}$$
$$= \frac{9\pi}{4} - \frac{18}{\pi^2}\cdot\frac{\pi^3}{24} = \frac{3\pi}{2}.$$

Choice of distractors for Group 1:

(d) If instead of subtracting the two integrals, one does the subtraction in the integrand before taking the square, one gets
$\frac{1}{2}\int_0^{\pi/2}(\rho_1(\varphi) - \rho_2(\varphi))^2 \,\mathrm{d}\varphi = \frac{1}{2}\int_0^{\pi/2}\left(9 - \frac{6\varphi}{\pi}\right)^2 \mathrm{d}\varphi = \frac{1}{2}\int_0^{\pi/2}\left(81 - \frac{108\varphi}{\pi} + \frac{36\varphi^2}{\pi^2}\right)\mathrm{d}\varphi = \frac{57\pi}{4}$.

(a) The attempt to calculate the area with elementary geometry and subtracting from the area of a quarter disc with radius 3 by mistake the area of a half disc of radius 1.5, one gets $\frac{1}{4}\cdot\pi\cdot 3^2 - \frac{1}{2}\cdot\pi\cdot 1.5^2 = \frac{9}{8}\pi$.

(c) If the area of the quarter disc is calculated without subtracting anything, one gets $\frac{9\pi}{4}$.